Assignment 3

Integration. Cauchy's Theorem

This assignment is shorter than usual, as most problems that I planned for it involve using the theorem that I didn't finish on the last lecture. This assignment will be therefore worth about 2/3 of others in terms of course grade.

I prefer that you submit this assignment by Wednesday, March 23rd. However, if you are somehow delayed, take your time (just don't get overwhelmed by homeworks piling up.)

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Prove that if $f(z)$ is continuous in the closed domain $|z| \geq R_0$, $0 \leq \arg z \leq \alpha$ $(0 \leq \alpha \leq 2\pi)$, and if the limit

$$\lim_{R\to\infty} zf(z) = 0$$

exists, then

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0,$$

where Γ_R is the arc of the circle $|z| = R$ lying in the given domain.

(2) Suppose that $f(z)$ is analytic in the closed domain $|z| \geq R_0$, $0 \leq \arg z \leq \alpha$ $(0 \leq \alpha \leq 2\pi)$, and if the limit $\lim_{R\to\infty} zf(z) = 0$ exists. Prove that if the integral

$$J_1 = \int_0^\infty f(x)dx$$

exists, then so does the integral

$$J_2 = \int_L f(z)dz,$$

where L is the ray $z = re^{i\alpha}$, $0 \leq r \leq \infty$. Moreover, show that $J_1 = J_2$. (Hint: use Cauchy's theorem and the problem above.)

(3) Evaluate the integral

$$\int_{|z-a|=R} (z-a)^n dz$$

$(R > 0)$ for all values of the integer n.

(4) Examine behavior of the integral

$$\int_{|z-i|=R} \frac{z^4+z^2+1}{z(z^2+1)}$$

as a function of $R > 0$. (Hint: $\frac{z^4+z^2+1}{z(z^2+1)} = z + \frac{1}{z} - \frac{1}{2}(\frac{1}{z+i} + \frac{1}{z-i})$.)

(5) Prove that

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi.$$

(Hint: integrate $\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta$ with $f(z) = \cos z$.)